

The sinh-Gordon defect matrix generalized for n defects

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Abstract In this paper we obtain a general expression for the n -defect matrix for the sinh-Gordon model. This in turn generate the general Bäcklund transformations (BT) for a system with n type-I defects, through a gauge transformation.

1 Introduction

Integrable models are known to be characterized by an infinite number of conservation laws which are responsible for the stability of soliton solutions. In fact, these conservation laws may be regarded as hamiltonians generating time evolutions within a multi-time space. Each of these time evolutions are associated to a non-linear equation of motion and henceforth constitute an integrable hierarchy of equations with common conservation laws. Another peculiar feature of integrable models is the existence of Bäcklund Transformations which relate two different field configurations of certain non-linear differential equation.

Bäcklund transformations (BT), among other applications, generate an infinite sequence of soliton solutions from a non-linear superposition principle (see [1]). These transformations have also been employed to describe integrable defects [2] in the sense that two solutions of an integrable model may be interpolated by a defect at certain spatial position. A BT connecting a two field configurations is the key ingredient to preserve the integrability of the system. Therefore, its systematic construction is important for the classification of integrable defects.

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The first type of Bäcklund transformation only involves the fields of the bulk theory, and is named type I. However, there exist integrable models for which such type of Bäcklund transformation are not allowed. This is the case of the Tzitzeica model where additional auxiliary fields are required [2, 3, 4, 5, 6]. These are called type II and consist of a new class of Bäcklund transformations. For the sine(sinh)-Gordon model, where type I Bäcklund transformation exists, the type II Bäcklund transformation is shown to be constructed from the composition of two type I defects.

The novelty presented in this paper is to extend the composition of several consecutive Gauge-Bäcklund transformations for the sinh-Gordon model. This provides the generalization to the case of n defects by constructing the general defect matrix, as well as the corresponding general BT. This is a powerful method since the defect matrix appears to be universal and can be used as a generator of BT for all equations within a hierarchy [7]. Finally, we will present few solutions for such composite BT.

2 Gauge-Bäcklund Transformation and Defect Matrices

The Lax pair for the sinh-Gordon model is given by

$$A_+(\phi_i) = \begin{pmatrix} \partial_+ \phi_i & 1 \\ \lambda & -\partial_+ \phi_i \end{pmatrix}, \quad A_-(\phi_i) = \begin{pmatrix} 0 & \frac{e^{-2\phi_i}}{\lambda} \\ e^{2\phi_i} & 0 \end{pmatrix}. \quad (1)$$

where we denote ϕ_0 and ϕ_1 to be solutions for $x < 0$ and $x > 0$ regions, respectively. The defect is placed at $x = 0$ and connects the two solutions by Bäcklund transformation. We assume the Lax pairs to be related by gauge transformation, i.e.,

$$K(\phi_0, \phi_1)A_\pm(\phi_1) = A_\pm(\phi_0)K(\phi_0, \phi_1) + \partial_\pm K(\phi_0, \phi_1) \quad (2)$$

where defect matrix describing the transition from solutions ϕ_0 to ϕ_1 is given by

$$K_i \equiv K(\phi_{i-1}, \phi_i) = \begin{pmatrix} 1 & -\frac{\sigma_i}{\lambda} e^{-(\phi_{i-1} + \phi_i)} \\ -\sigma_i e^{(\phi_{i-1} + \phi_i)} & 1 \end{pmatrix}. \quad (3)$$

and σ_i is the corresponding Bäcklund parameter. The gauge transformation (2) holds provided the following first order equations are satisfied,

$$\partial_+(\phi_0 - \phi_1) = -2\sigma_1 \sinh(\phi_0 + \phi_1), \quad \text{and} \quad \partial_-(\phi_0 + \phi_1) = -\frac{2}{\sigma_1} \sinh(\phi_0 - \phi_1), \quad (4)$$

where $\partial_\pm = \frac{1}{2}(\partial_x \pm \partial_t)$. Equations (4) are the type I Bäcklund transformations for the sinh-Gordon model.

Let us now consider the composition of two Bäcklund-gauge transformations $K^{(2)}(\phi_0, \phi_2) = K(\phi_1, \phi_2)K(\phi_0, \phi_1)$. From expression (3) we find

$$K^{(2)} = \begin{pmatrix} 1 + \frac{\sigma_1 \sigma_2}{\lambda} e^{p_1 - p_2} & -\frac{1}{\lambda} (\sigma_1 e^{-p_1} + \sigma_2 e^{-p_2}) \\ -\sigma_1 e^{p_1} - \sigma_2 e^{p_2} & 1 + \frac{\sigma_1 \sigma_2}{\lambda} e^{-p_1 + p_2} \end{pmatrix}. \quad (5)$$

Denoting $\eta = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1 \sigma_2}$, $\sigma^2 = -\frac{1}{\sigma_1 \sigma_2}$ and defining

$$\Lambda = -\phi_1 - \ln(2\sigma_2 e^{-\phi_0} + 2\sigma_1 e^{-\phi_2}) - \ln \frac{\sigma}{4}, \quad (6)$$

we obtain the type II Bäcklund transformations proposed in [5, 8], namely,

$$K^{(2)}(p, q, \Lambda) = \begin{pmatrix} 1 - \frac{1}{\sigma^2 \lambda} e^q & \frac{e^{\Lambda - p}}{2\lambda \sigma} (e^q + e^{-q} + \eta) \\ -\frac{2}{\sigma} e^{p - \Lambda} & 1 - \frac{1}{\lambda \sigma^2} e^{-q} \end{pmatrix}, \quad (7)$$

where $q = \phi_0 - \phi_2$, $p = \phi_0 + \phi_2$.

Now we consider a system with n Type-I defects, each one with a different parameter σ_i as showing in the following the diagram,

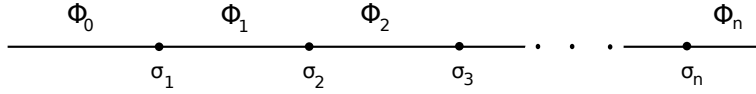


Fig. 1 Generalization for n type-I defects.

By defining $K^{(n)}$ in the following form,

$$K^{(n)} = K_n K_{n-1} \dots K_2 K_1 = \begin{pmatrix} K_{11}^{(n)} & K_{12}^{(n)} \\ K_{21}^{(n)} & K_{22}^{(n)} \end{pmatrix}, \quad (8)$$

we find for even n :

$$\begin{aligned} K_{11} &= 1 + \left[\prod_{a=1}^{n_\sigma} \left(\sum_{i_a=a}^{n-(n_\sigma-a)} \right) \right] \left[\sum_{r=1}^{n/2} \frac{1}{\lambda^r} \prod_{j=1}^{2r} \sigma_{i_j} e^{(-1)^{j+1} p_{i_j}} \right], \\ K_{12} &= - \left[\prod_{a=1}^{n_\sigma} \left(\sum_{i_a=a}^{n-(n_\sigma-a)} \right) \right] \left[\sum_{r=0}^{(n-2)/2} \frac{1}{\lambda^{r+1}} \prod_{j=1}^{2r+1} \sigma_{i_j} e^{(-1)^j p_{i_j}} \right], \\ K_{21} &= - \left[\prod_{a=1}^{n_\sigma} \left(\sum_{i_a=a}^{n-(n_\sigma-a)} \right) \right] \left[\sum_{r=0}^{(n-2)/2} \frac{1}{\lambda^r} \prod_{j=1}^{2r+1} \sigma_{i_j} e^{(-1)^{j+1} p_{i_j}} \right], \\ K_{22} &= 1 + \left[\prod_{a=1}^{n_\sigma} \left(\sum_{i_a=a}^{n-(n_\sigma-a)} \right) \right] \left[\sum_{r=1}^{n/2} \frac{1}{\lambda^r} \prod_{j=1}^{2r} \sigma_{i_j} e^{(-1)^j p_{i_j}} \right], \end{aligned} \quad (9)$$

and for odd n :

$$\begin{aligned}
K_{11} &= 1 + \left[\prod_{a=1}^{n_\sigma} \binom{n-(n_\sigma-a)}{\sum_{i_a=a}} \right] \left[\sum_{r=1}^{(n-1)/2} \frac{1}{\lambda^r} \prod_{j=1}^{2r} \sigma_{i_j} e^{(-1)^{j+1} p_{i_j}} \right], \\
K_{12} &= - \left[\prod_{a=1}^{n_\sigma} \binom{n-(n_\sigma-a)}{\sum_{i_a=a}} \right] \left[\sum_{r=0}^{(n-1)/2} \frac{1}{\lambda^{r+1}} \prod_{j=1}^{2r+1} \sigma_{i_j} e^{(-1)^j p_{i_j}} \right], \\
K_{21} &= - \left[\prod_{a=1}^{n_\sigma} \binom{n-(n_\sigma-a)}{\sum_{i_a=a}} \right] \left[\sum_{r=0}^{(n-1)/2} \frac{1}{\lambda^r} \prod_{j=1}^{2r+1} \sigma_{i_j} e^{(-1)^{j+1} p_{i_j}} \right], \\
K_{22} &= 1 + \left[\prod_{a=1}^{n_\sigma} \binom{n-(n_\sigma-a)}{\sum_{i_a=a}} \right] \left[\sum_{r=1}^{(n-1)/2} \frac{1}{\lambda^r} \prod_{j=1}^{2r} \sigma_{i_j} e^{(-1)^j p_{i_j}} \right], \quad (10)
\end{aligned}$$

where n_σ is the number of parameters σ_{i_j} associated with each defect such that $i_1 < i_2 < i_3 < \dots < i_n$, and $p_{i_j} = \phi_{i_j} + \phi_{i_j-1}$.

The next step is to derive a general expression for the BT corresponding to this $K^{(n)}$ defect matrix. In order to obtain the Bäcklund transformations for n defects, we will assume $K^{(n)}$ to be the generator of the gauge transformation (2), leading to

$$\begin{aligned}
\partial_+(\phi_0 - \phi_n) &= -2 \sum_{i=1}^n \sigma_i \sinh p_i \\
\partial_-(\phi_0 - (-1)^n \phi_n) &= 2 \sum_{i=1}^n \frac{(-1)^n}{\sigma_i} \sinh q_i \\
\partial_+ q_i &= -2 \sigma_i \sinh p_i \\
\partial_- p_i &= -\frac{2}{\sigma_i} \sinh q_i \quad (11)
\end{aligned}$$

with $p_i = \phi_{i-1} + \phi_i$, $q_i = \phi_{i-1} - \phi_i$, and $i = 1, \dots, n$.

3 Bäcklund solutions

In this section we will consider some solutions of the sinh-Gordon model in the presence of two and three defects.

$\mathbf{n = 2}$

Consider now the fields ϕ_0 and ϕ_2 on each side of the defect with an intermediary field ϕ_1 ,

Vacuum \rightarrow One Soliton \rightarrow Vacuum Solution. First of all, we consider the following solution:

$$\phi_0 = 0, \quad \phi_2 = 0, \quad \phi_1 = \ln \left(\frac{1 + \rho_1}{1 - \rho_1} \right), \quad \rho_1 = \exp \left(2k_1 x_+ + \frac{2}{k_1} x_- \right), \quad (12)$$

which satisfy the Bäcklund equations (11) with $n = 2$ with the following conditions

$$\sigma_1 = k_1, \quad \sigma_2 = -k_1. \quad (13)$$

Vacuum \rightarrow One Soliton \rightarrow Two Soliton Solution. Another possible solution is

$$\begin{aligned} \phi_0 = 0, \quad \phi_1 = \ln \left(\frac{1 + \rho_1}{1 - \rho_1} \right), \quad \phi_2 = \ln \left(\frac{1 + b_1 \rho_1 + b_2 \rho_2 + \alpha_{12} b_1 b_2 \rho_1 \rho_2}{1 - b_1 \rho_1 - b_2 \rho_2 + \alpha_{12} b_1 b_2 \rho_1 \rho_2} \right), \\ \rho_j = \exp \left(2k_j x_+ + \frac{2}{k_j} x_- \right), \quad j = 1, 2 \end{aligned} \quad (14)$$

where, in order to ϕ_2 satisfies the sinh-Gordon equation $\alpha_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2$. Analogously, we get the following Bäcklund conditions:

$$\sigma_1 = k_1, \quad \sigma_2 = k_2, \quad b_1 = \frac{k_1 + k_2}{k_1 - k_2}. \quad (15)$$

n = 3

Finally putting a third defect at the same point of the others we have the fields ϕ_0 and ϕ_3 on each side of the defects and two intermediary fields ϕ_1 and ϕ_2 at the defect points.

Vacuum \rightarrow One Soliton \rightarrow Vacuum \rightarrow One Soliton Solution. Now taking into account the solutions:

$$\phi_0 = 0, \quad \phi_2 = 0, \quad \phi_1 = \ln \left(\frac{1 + \rho_1}{1 - \rho_1} \right), \quad \phi_3 = \ln \left(\frac{1 + \rho_2}{1 - \rho_2} \right) \quad (16)$$

The Bäcklund conditions in order to satisfy the Type-II BT are: $\sigma_1 = k_1, \sigma_2 = -k_1, \sigma_3 = k_2$.

Vacuum \rightarrow One Soliton \rightarrow Two Soliton \rightarrow Three Soliton Solution. Lastly, we assume:

$$\begin{aligned} \phi_0 = 0, \quad \phi_1 = \ln \left(\frac{1 + \rho_1}{1 - \rho_1} \right), \quad \phi_2 = \ln \left(\frac{1 + b_1 \rho_1 + b_2 \rho_2 + \alpha_{12} b_1 b_2 \rho_1 \rho_2}{1 - b_1 \rho_1 - b_2 \rho_2 + \alpha_{12} b_1 b_2 \rho_1 \rho_2} \right), \\ \phi_3 = \ln \left(\frac{1 + R_1 + R_2 + R_3 + \alpha_{12} R_1 R_2 + \alpha_{13} R_1 R_3 + \alpha_{23} R_2 R_3 + \alpha_{123} R_1 R_2 R_3}{1 - R_1 - R_2 - R_3 + \alpha_{12} R_1 R_2 + \alpha_{13} R_1 R_3 + \alpha_{23} R_2 R_3 - \alpha_{123} R_1 R_2 R_3} \right), \\ \rho_j = \exp \left(2k_j x_+ + \frac{2}{k_j} x_- \right), \quad R_j = a_j \rho_j, \quad j = 1, 2, 3 \end{aligned} \quad (17)$$

where in order to ϕ_2 and ϕ_3 satisfy the sinh-Gordon equation we should have

$$\begin{aligned} \alpha_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2, \quad \alpha_{23} = \left(\frac{k_2 - k_3}{k_2 + k_3} \right)^2, \quad \alpha_{13} = \left(\frac{k_1 - k_3}{k_1 + k_3} \right)^2, \\ \alpha_{123} = \alpha_{12} \alpha_{13} \alpha_{23}. \end{aligned} \quad (18)$$

In this case the Bäcklund conditions are: $\sigma_1 = k_1$, $\sigma_2 = k_2$, $\sigma_3 = k_3$, $b_1 = \frac{k_1+k_2}{k_1-k_2}$, $a_1 = \left(\frac{k_1+k_3}{k_1-k_3}\right)b_1$, and $a_2 = \left(\frac{k_2+k_3}{k_2-k_3}\right)b_2$, where b_2 is a free parameter. It is worth mentioning that the BT and their solutions for a four-defect system have been also computed, and the results have shown the expected behaviour.

4 Conclusion

In this paper we have considered the sinh-Gordon model and provided general formulas for the defect matrix when n defects are considered. Our construction involves the product of n Type-I defect matrices. In addition, we have calculated their respective BT in a general way through gauge transformations and provided a few simple examples for $n = 2, 3$.

It is important to point out that, since the BT are constructed as gauge transformations, they preserve the zero curvature representation. The latter describes a hierarchy of integrable equations based upon an universal Lax operator. These two facts induce the idea of the universality of the Bäcklund-Gauge transformation within the hierarchy. We have verified [9, 10] that the constructed defect matrix indeed gives the correct BT for the mKdV equation. It provides a systematic construction of BT for all higher grade evolution equations within the mKdV hierarchy. Several examples were verified for KdV hierarchies [7] as well.

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